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Minimum uncertainty states for the quantum group $SU_q(2)$ and quantum Wigner d-functions

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Abstract. Minimum uncertainty angular momentum states for the quantum group $SU_q(2)$ are constructed. They involve the eigenvalues of J_1 which are q-numbers and the quantum group analogue of the Wigner d-functions for $\theta = \pi/2$. The result is generalized for all values of θ and a formula for the quantum Wigner d-function is derived. The case of $q = 1$ is discussed and compared with the well known results for the Wigner d-functions.

1. Introduction

Quantum groups originate in the study of quantum inverse scattering methods and since then there have been many extensive studies on their representation theory and structure. In this paper we discuss the finite dimensional representation of the quantum group $SU_q(2)$ with the intention of constructing minimum uncertainty states. In the process we are led to the notion of rotation in quantum groups and the matrix elements of angular momentum operators. These define the quantum group analogue of Wigner d-functions (rotation matrices) which reduce to the well known $d^j_{mm'}(\theta)$ functions when $q = 1$. A result for the ratio of two such quantum Wigner d-functions as a continued fraction is obtained. This is verified to be true for $q = 1$. For $q \neq 1$, it is suggestive from our discussion that the quantum Wigner d-function can be expressed in terms of a ${}_2\phi_1$ q-hypergeometric function.

It is well known that the representation theory of classical Lie algebras gives an algebraic setting for many special functions in mathematical physics. Quantum (deformed) Lie groups and algebras play a role for generalization to base q (the deformation parameter) of these functions, namely, q-special functions. Here we motivate one such q-function, *a q-hypergeometric function* ${}_2\phi_1$ (Heine's series) by considering rotation in $SU_q(2)$.

The coherent states of the quantum Heisenberg group formed by q-boson creation, annihilation and the number operators satisfy the minimum uncertainty property [1]. On the other hand it is known that in the case of ordinary $SU(2)$, the angular momentum coherent states do not necessarily satisfy the minimum uncertainty property [2]. The angular momentum coherent states and those that give minimum uncertainty product are respectively called Bloch and 'intelligent' states in the literature. Explicit construction procedures for them in the case of ordinary angular momentum algebra are known [2]. For the quantum group $SU_q(2)$, coherent states have been constructed by Jurco [3] and recently by Aref'eva *et al* [4] using the theory of harmonic analysis. In this paper, we construct states having the

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minimum uncertainty for $SU_q(2)$ by introducing the notion of rotation in quantum groups and give the matrix elements of the rotation operators.

2. Construction of minimum uncertainty states

The q-deformed algebra of $SU_q(2)$ is an associative algebra of the generators $J_3, J_\pm (= J_1 \pm \mathrm{i} J_2)$ satisfying

$$\begin{aligned}&[J_3, J_\pm] = \pm J_\pm \\ &[J_+, J_-] = [2J_3]\end{aligned} \tag{1}$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The irreducible representation of $SU_q(2)$ in a $(2j+1)$-dimensional Hilbert space is spanned by vectors $|j, -j\rangle, |j, -j+1\rangle, |j, -j+2\rangle, \ldots, |j, j\rangle$ and the action of the generators (1) is

$$\begin{aligned}&J_3|j,m\rangle = m|j,m\rangle \\ &J_+|j,m\rangle = ([j-m][j+m+1])^{1/2}|j,m+1\rangle \\ &J_-|j,m\rangle = ([j+m][j-m+1])^{1/2}|j,m-1\rangle.\end{aligned} \tag{2}$$

Jurco [3] defined the coherent state for $SU_q(2)$ as

$$|\alpha\rangle = \mathrm{e}_q^{\alpha J_+}|j, -j\rangle \tag{3}$$

where e_q is q-exponential function,

$$\mathrm{e}_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}.$$

The state in (3) is (i) continuous in α and (ii) satisfies the completeness property (resolution of unity). The uncertainty relation following from (1) is

$$\triangle J_1 \cdot \triangle J_2 \geqslant \tfrac{1}{4}|\langle[2J_3]\rangle| \tag{4}$$

and it can be verified that the coherent state (3) does not have the minimum uncertainty property. An important observation regarding (3) is worth pointing out. In the ordinary $SU(2)$ case, we have $\mathrm{e}^{\alpha J_+}|j, -j\rangle$ which amounts to a mere redefinition of the axes of the system and so it is a trivial choice. However, in the quantum $SU_q(2)$ case, as in (3), we have a q-exponential function which is not a rotation operator in the standard sense and so (3) is a non-trivial choice. In fact a proper representation of a rotation operator for this quantum group will be considered in due course.

We exploit the first relation in (1) in constructing minimum uncertainty state for $SU_q(2)$. We modify the well known procedure [5] in the ordinary $SU(2)$ case as below. It can be shown [5] that the minimum uncertainty states of the angular momentum algebra (1) satisfy

$$(J_1 + \mathrm{i}\lambda J_2)|\psi\rangle = \mu|\psi\rangle. \tag{5}$$

To solve this equation, we need fortunately the first relation in (1) alone. Let $\lambda = \tanh\beta$. Then (5) becomes

$$(\mathrm{e}^{\beta} J_+ + \mathrm{e}^{-\beta} J_-)|\psi\rangle = \mu'|\psi\rangle \tag{6}$$

where $\mu' = \mu(\mathrm{e}^{\beta} + \mathrm{e}^{-\beta})$. Due to the first relation in (1), we have $\mathrm{e}^{\beta J_3} J_\pm \mathrm{e}^{-\beta J_3} = \mathrm{e}^{\pm\beta} J_\pm$. (Note that we have the ordinary exponential here.) Therefore we get

$$\mathrm{e}^{\beta J_3}(J_+ + J_-)\mathrm{e}^{-\beta J_3}|\psi\rangle = \mu'|\psi\rangle$$

and in the usual way [5], the solution has the following form:

$$|\psi\rangle = N e^{\beta J_3} |\phi\rangle \tag{7}$$

where $|\phi\rangle$ is an eigenstate of J_1 and N is the normalization constant given for $q \neq 1$ case as

$$N = \left\{ \sum_{m=-j}^{j} e^{2\beta m} (C^j_{m,\lambda(\tilde{m})})^2 \right\}^{-1/2}$$

where $C^j_{m,\lambda(\tilde{m})}$ will be determined shortly.

The crucial point is that in the case of ordinary $SU(2)$, $|\phi\rangle$ is simply $e^{-i\frac{\pi}{2}J_2}|j,m\rangle$ while now such a simple result is not possible. We now proceed to construct $|\phi\rangle$ explicitly for $SU_q(2)$. $|\phi\rangle$ is an eigenstate of J_1 with eigenvalue $\lambda(\tilde{m})$ (say). $\tilde{m}$ is just a running label ranging from $-j$ to j and $\lambda(\tilde{m})$ is a q-number. $|\phi\rangle$ is expanded in terms of the eigenstates $|j,m\rangle$ of J_3 with coefficients $C^j_{m,\lambda(\tilde{m})}$. These coefficients when $q = 1$ are the usual Wigner d-functions with θ equal to $\pi/2$. So,

$$|\phi\rangle = |j, \lambda(\tilde{m})\rangle_1 = \sum_{m=-j}^{j} C^j_{m,\lambda(\tilde{m})} |j,m\rangle \tag{8}$$

where the subscript 1 denotes eigenstate of J_1. The coefficients $C^j_{m,\lambda(\tilde{m})}$ and the eigenvalues $\lambda(\tilde{m})$ are to be determined. Since $|j,m\rangle$ in (7) are eigenstates of J_3, it follows from (2), that

$$\begin{aligned} J_1|j,\lambda(\tilde{m})\rangle_1 = \tfrac{1}{2} \sum_{m=-j}^{j} \{ & C^j_{(m+1),\lambda(\tilde{m})}([j+m+1][j-m])^{1/2} \\ & + C^j_{(m-1),\lambda(\tilde{m})}([j-m+1][j+m])^{1/2}\} |j,m\rangle. \end{aligned} \tag{9}$$

In order that $|j,\lambda(\tilde{m})\rangle_1$ be an eigenstate of J_1 with eigenvalues $\lambda(\tilde{m})$, it follows from (8) and (9) that,

$$\begin{aligned} C^j_{(m+1),\lambda(\tilde{m})}([j+m+1][j-m])^{1/2} + C^j_{(m-1),\lambda(\tilde{m})}([j-m+1][j+m])^{1/2} \\ = 2\lambda(\tilde{m}) C^j_{m,\lambda(\tilde{m})}. \end{aligned} \tag{10}$$

This is our 'master equation', determining $C^j_{m,\lambda(\tilde{m})}$ and $\lambda(\tilde{m})$. By setting $m = j, j-1, j-2$ and so on, we find

$$\begin{aligned} C^j_{(j-1),\lambda(\tilde{m})} &= \frac{2\lambda(\tilde{m})}{\sqrt{[2j]}} C^j_{j,\lambda(\tilde{m})} \\ C^j_{(j-2),\lambda(\tilde{m})} &= \frac{4\lambda^2(\tilde{m}) - [2j]}{\sqrt{([2][2j][2j-1])}} C^j_{j,\lambda(\tilde{m})} \\ C^j_{(j-3),\lambda(\tilde{m})} &= \frac{2\lambda(\tilde{m})(4\lambda^2(\tilde{m}) - [2j] - [2][2j-1])}{\sqrt{([2][3][2j][2j-1][2j-2])}} C^j_{j,\lambda(\tilde{m})} \\ C^j_{(j-4),\lambda(\tilde{m})} &= \frac{16\lambda^4(\tilde{m}) - 4\lambda^2(\tilde{m})([2j] + [2][2j-1] + [3][2j-2]) + [3][2j][2j-2]}{\sqrt{([2][3][4][2j][2j-1][2j-2][2j-3])}} \\ &\quad \times C^j_{j,\lambda(\tilde{m})} \end{aligned} \tag{11}$$

and so on. In the above $C^j_{j,\lambda(\tilde{m})}$ can be considered as an overall multiplicative factor which can be included in the normalization of $|\phi\rangle$ and then the above equations give the expansion coefficients. For a given j, we equate $C^j_{(-j-1),\lambda(\tilde{m})}$ to zero to determine $\lambda(\tilde{m})$. It is obvious that the eigenvalues $\lambda(\tilde{m})$ come in pairs with opposite sign.

We now give a closed expression for $C^j_{m,\lambda(\tilde{m})}$. Denoting $2\lambda(\tilde{m})$ by a and $[p][2j-(p-1)]$ by b_p and the 'continued fraction',

$$a - \frac{b_p}{a-}\frac{b_{p-1}}{a-}\frac{b_{p-2}}{a-}\frac{b_{p-3}}{a-}\cdots\frac{b_1}{a} \tag{12}$$

by F_p, equation (11) can be rewritten as

$$\sqrt{([p+1][2j-p])}C^j_{j-(p+1),\lambda(\tilde{m})} = F_p C^j_{j-p,\lambda(\tilde{m})} \tag{13}$$

for $p = 0$ to $2j-1$. The above result allows us to express any two C-functions differing by unity in the first index in the subscript for a given $\lambda(\tilde{m})$. From this it is possible to express $C^j_{m,\lambda(\tilde{m})}$ in terms of $C^j_{j,\lambda(\tilde{m})}$ for $m = -j$ to j as

$$\left\{\prod_{\ell=1}^{j-m}[\ell][2j-(\ell-1)]\right\}^{1/2} C^j_{m,\lambda(\tilde{m})} = \left\{\prod_{\ell=0}^{j-m-1} F_\ell\right\} C^j_{j,\lambda(\tilde{m})} \tag{14}$$

for $m \neq j$. Therefore, the eigenstate of J_1 in equation (8) becomes

$$|\phi\rangle = C^j_{j,\lambda(\tilde{m})}\sum_{m=-j}^{j}\frac{\prod_{\ell=0}^{j-m-1}F_\ell}{\{\prod_{\ell=1}^{j-m}[\ell][2j-(\ell-1)]\}^{1/2}}|j,m\rangle. \tag{15}$$

The multiplicative factor $C^j_{j,\lambda(\tilde{m})}$ in the above equation is given by

$$C^j_{j,\lambda(\tilde{m})} = \left(\sum_{m=-j}^{j}\frac{\prod_{\ell=0}^{j-m-1}F_\ell^2}{\prod_{\ell=1}^{j-m}[\ell][2j-(\ell-1)]}\right)^{-1/2}.$$

This completes the construction of minimum uncertainty states for quantum group $SU_q(2)$.

2.1. Illustrations

We illustrate the above procedure for the construction of minimum uncertainty states for $SU_q(2)$ with $j = 1, 2$. It consists of two steps; first finding the eigenvalues $\lambda(\tilde{m})$ of J_1 operator with $\tilde{m}$ eigenvalues of J_3 i.e. $-j$ to $+j$. For $j = 1$, these are determined by equating $C^1_{-2,\lambda(\tilde{m})}$ to zero. They are given by

$$\lambda(0) = 0 \qquad \lambda(1) = \sqrt{\left(\frac{[2]}{2}\right)} \qquad \lambda(-1) = -\sqrt{\left(\frac{[2]}{2}\right)}. \tag{16}$$

Having determined the eigenvalues $\lambda(\tilde{m})$, we evaluate the C-functions using the 'master equation' (10) for general $\lambda(\tilde{m})$ as

$$C^1_{0,\lambda(\tilde{m})} = \frac{2\lambda(\tilde{m})}{\sqrt{[2]}}C^1_{1,\lambda(\tilde{m})}$$
$$C^1_{-1,\lambda(\tilde{m})} = \left(\frac{4\lambda(\tilde{m})^2}{[2]} - 1\right)C^1_{1,\lambda(\tilde{m})}. \tag{17}$$

The above equations give the C-functions for specific values of $\lambda(\tilde{m})$ in (16) in terms of $C^1_{1,\lambda(\tilde{m})}$.

A similar procedure applied to $j = 2$, yields the following $\lambda(\tilde{m})$:

$$\lambda(0) = 0 \qquad \lambda(\pm 1) = \pm\frac{[4]^{1/2}}{2} \qquad \lambda(\pm 2) = \pm\tfrac{1}{2}\{[4] + 2[2][3]\}^{1/2}. \tag{18}$$

It can be verified that when $q = 1$, we obtain the standard eigenvalues. The C-functions can be determined by once again using (10) and these are in general given by

$$C^2_{1,\lambda(\tilde{m})} = \frac{2\lambda(\tilde{m})}{\sqrt{[4]}} C^2_{2,\lambda(\tilde{m})}$$
$$C^2_{0,\lambda(\tilde{m})} = \frac{(4\lambda(\tilde{m})^2 - [4]}{\sqrt{([2][3][4])}} C^2_{2,\lambda(\tilde{m})}$$
$$C^2_{-1,\lambda(\tilde{m})} = \frac{2\lambda(\tilde{m})\{4\lambda(\tilde{m})^2 - [4] - [2][3]\}}{[2][3]\sqrt{[4]}} C^2_{2,\lambda(\tilde{m})}$$
$$C^2_{-2,\lambda(\tilde{m})} = \left(\frac{4\lambda(\tilde{m})^2(4\lambda(\tilde{m})^2 - [4] - [2][3])}{[2][3][4]} - \frac{4\lambda(\tilde{m})^2 - [4]}{[4]}\right) C^2_{2,\lambda(\tilde{m})}. \tag{19}$$

The above expressions give the required C-functions in terms of $C^2_{2,\lambda(\tilde{m})}$ for various $\lambda(\tilde{m})$ in (18).

We now give the minimum uncertainty angular momentum $SU_q(2)$ states for $j = 1$ and 2. As can be seen from (7) this depends upon the choice of specific value for $\lambda(\tilde{m})$, the eigenvalue of the operator J_1. As an example first we consider $j = 1$ with $\lambda(\tilde{m}) = \sqrt{[2]}/\sqrt{2}$. Then (7) gives the minimum uncertainty angular momentum state as

$$|\beta\rangle = \frac{1}{2N}\{e^{\beta}|1, 1 > +\sqrt{2}|1, 0\rangle + e^{-\beta}|1, -1\rangle\} \tag{20}$$

where $N = \frac{1}{2}(e^{\beta} + e^{-\beta})$. For this state, the variances are found to be

$$(\triangle J_1)^2 = \frac{[2]}{16N^2}(e^{\beta} - e^{-\beta})^2$$
$$(\triangle J_2)^2 = \frac{[2]}{16N^2}(e^{\beta} + e^{-\beta})^2 \tag{21}$$

and

$$|\langle[2J_3]\rangle|^2 = \frac{[2]^2}{16N^4}(e^{2\beta} - e^{-2\beta})^2 \tag{22}$$

satisfying $(\triangle J_1)^2 \cdot (\triangle J_2)^2 = \frac{1}{16}|\langle[2J_3]\rangle|^2$ and $\triangle J_1 \neq \triangle J_2$.

A mimimum uncertainty state for $j = 2$ can similarly be constructed by using the eigenvalues in (18). Choosing the zero eigenvalue for $\lambda(\tilde{m})$ as an example, a minimum uncertainty state for $j = 2$ is given by

$$|\beta\rangle = N'\{e^{-2\beta}|2, -2\rangle - \frac{[4]}{\sqrt{([2][3][4])}}|2, 0\rangle + e^{2\beta}|2, 2\rangle\}. \tag{23}$$

The variances are evaluated as

$$(\triangle J_1)^2 = \frac{N'^2}{4}[4](2 + e^{4\beta} + e^{-4\beta} - 2e^{-2\beta} - 2e^{2\beta})$$
$$(\triangle J_2)^2 = \frac{N'^2}{4}[4](2 + e^{4\beta} + e^{-4\beta} + 2e^{-2\beta} + 2e^{2\beta})$$
$$\langle[2J_3]\rangle = N'^2[4](e^{4\beta} - e^{-4\beta}). \tag{24}$$

It can be seen that $(\triangle J_1)^2 \cdot (\triangle J_2)^2 = \frac{1}{16}(\langle[2J_3]\rangle)^2$, and $\triangle J_1 \neq \triangle J_2$.

To summarize this section, we have explicitly constructed minimum uncertainty states for the quantum group $SU_q(2)$. These involve the eigenvalues of J_1 which are q-numbers and the coefficients $C^j_{m,\lambda(\tilde{m})}$. These coefficients are the quantum group analogues of the Wigner $d^j_{m,\tilde{m}}$ functions for $\theta = \pi/2$. Examples are given for $j = 1$ and 2 for chosen $\lambda(\tilde{m})$. This can be easily carried over to other values and other j values as well. It can be seen that the minimum uncertainty states coincide with those for ordinary $SU(2)$ only when $q = 1$.

3. Quantum Wigner C-functions

The Wigner d-functions $d^j_{mm'}(\theta)$ play an important role in the theory of representations of the group $SU(2)$. In the literature [6] these functions are usually given in terms of ${}_2F_1$-hypergeometric functions or equivalently as a finite series. The Wigner d-functions play a dual role. To see this, let us consider rotation about the y axis. The rotation operator is given by

$$R_y(\theta) = \mathrm{e}^{-\mathrm{i}\theta J_2} \tag{25}$$

and then

$$d^j_{mm'}(\theta) = \langle jm|\mathrm{e}^{-\mathrm{i}\theta J_2}|jm'\rangle. \tag{26}$$

The operator $R_y(\theta)$ rotates J_3 to $J'_3 \equiv J_3\cos\theta + J_1\sin\theta$ and the matrix elements of J'_3 are

$$(J'_3)_{mm'} = \sum_{p,r,m=-j}^{j} d^j_{mp}(\theta)(J_3)_{pr} d^{\dagger j}_{rm}(\theta). \tag{27}$$

Here Wigner d-functions relate the matrix elements of J_3 and J'_3. Relatedly, the basis in which J'_3 is diagonal can be written as

$$|jm'\rangle_{J'_3} = \sum_{m=-j}^{j} d^j_{mm'}(\theta)|jm\rangle_{J_3}. \tag{28}$$

These two roles of the Wigner d-functions have been used by Fano and Racah [7] to facilitate the *determination* of $d^j_{mm'}(\theta)$. It is worth recalling that the eigenvalues of the operators J_1, J_2 and J_3 form the set $\{-j, -j+1, \ldots, +j\}$ and this is due to the fact that these operators are unitarily equivalent. In the construction of minimum uncertainty states in section 2, we required the eigenvalues of J_1 and these are readily provided by the above equation for $\theta = \pi/2$ for ordinary $SU(2)$. The case of the quantum group $SU_q(2)$ as far as the rotation operators are concerned is quite different as the operators J_1 (or J_2) and J_3 are *not* equivalent and they do not appear symmetrically in the q-deformed algebra (1). The construction of minimum uncertainty states for $SU_q(2)$ required the quantum group analogue of the Wigner d-functions, denoted by $C^j_{m,\lambda(\tilde{m})}(\pi/2)$ and these are constructed in section 2. In this section, we define the notion of rotation in $SU_q(2)$ and give a representation for quantum Wigner functions in terms of continued fractions.

While the meaning of a rotation by an angle θ around, say, the y axis for ordinary $SU(2)$ is well known, it is not so for $SU_q(2)$ as the q-exponential fails to satisfy the desired properties. To motivate a definition of rotation for $SU_q(2)$, we note that for ordinary $SU(2)$ the above-mentioned rotation takes J_3 to J'_3:

$$J'_3 = \cos\theta J_3 + \sin\theta J_1. \tag{29}$$

Alternately, the same transformation applied to a basis in the ordinary $SU(2)$ in which J_3 is diagonal results in a basis in which J'_3 (29) is diagonal. *We take the later as our definition for a rotation by θ around the y axis in $SU_q(2)$.* To be consistent with q-functions, the ordinary trignometric functions $\sin\theta$ and $\cos\theta$ will be replaced by the corresponding pseudo-periodic q-circular functions $S_q(\theta)$ and $C_q(\theta)$ respectively. These q-circular functions have been introduced by Jackson [8] with quasi-period $\omega = (\Gamma_q(1/2))^2$, in terms of q-Gamma function. In the limit $q = 1$, $\omega \to \pi$ and $S_q(\theta)$ and $C_q(\theta)$ go over to $\sin(\theta)$ and $\cos(\theta)$, respectively. Further properties of these q-circular functions are given in [9]. Thus, a rotation by θ around the y axis takes J_3 to J'_3 and is given by

$$J'_3 = C_q(\theta)J_3 + S_q(\theta)J_1. \tag{30}$$

Denoting the $(2j+1)$ eigenvalues of J_3' by $\lambda(\tilde{m},\theta)$ (where $\tilde{m} = -j$ to j; and $\lambda(\tilde{m},\theta)$ are in general q-functions of $S_q(\theta)$ and $C_q(\theta)$) and writing an eigenstate of J_3' as $|j,\lambda(\tilde{m},\theta)\rangle$, we have,

$$J_3'|j,\lambda(\tilde{m},\theta)\rangle = \lambda(\tilde{m},\theta)|j,\lambda(\tilde{m},\theta)\rangle. \tag{31}$$

Now expanding $|j,\lambda(\tilde{m},\theta)\rangle$ in terms of the eigenstates of J_3, we have

$$|j,\lambda(\tilde{m},\theta)\rangle = \sum_{m=-j}^{j} C^j_{m,\lambda(\tilde{m},\theta)}|j,m\rangle \tag{32}$$

where the expansion coefficients $C^j_{m,\lambda(\tilde{m},\theta)}$ are the quantum group $SU_q(2)$ analogue of the Wigner d-functions for all θ values. Using the actions of J_3, $J_\pm$ on $|j,m\rangle$ as in (2), we find the following recursion relation for $C^j_{m,\lambda(\tilde{m},\theta)}$:

$$\begin{aligned}([j+m][j-m+1])^{1/2}S_q(\theta)C^j_{m-1,\lambda(\tilde{m},\theta)}(\theta)&+([j-m][j+m+1])^{1/2}S_q(\theta)C^j_{m+1,\lambda(\tilde{m},\theta)}(\theta)\\ &= 2(\lambda(\tilde{m},\theta)-mC_q(\theta))C^j_{m,\lambda(\tilde{m},\theta)}(\theta).\end{aligned} \tag{33}$$

Defining a_m and b_p as

$$\begin{aligned}a_m &= 2(\lambda(\tilde{m},\theta)-mC_q(\theta))\\ b_p &= [p][2j-(p-1)]S_q^2(\theta)\end{aligned} \tag{34}$$

we can in general relate two C-functions differing by one in the first subscript, namely, the eigenvalue of J_3, in terms of a *continued fraction* as

$$\begin{aligned}\{[p+1][2j-p]\}^{1/2}S_q(\theta)C^j_{j-(p+1),\lambda(\tilde{m},\theta)}(\theta)\\ = \left\{a_{j-p} - \frac{b_p}{a_{j-(p-1)}-}\frac{b_{p-1}}{a_{j-(p-2)}-}\frac{b_{p-2}}{a_{j-(p-3)}-}\cdots\frac{b_1}{a_j}\right\}C^j_{j-p,\lambda(\tilde{m},\theta)}(\theta).\end{aligned} \tag{35}$$

As in section 2, the above relation (35) plays two roles. Firstly, for a given j, equating $C^j_{-j-1,\lambda(\tilde{m},\theta)}(\theta)$ to zero, the eigenvalues $\lambda(\tilde{m},\theta)$ are determined. Secondly, using these eigenvalues, the quantum Wigner functions can all be determined in terms of $C^j_{j,\lambda(\tilde{m},\theta)}(\theta)$ which itself can be determined by normalization of (32) as in section 2.

Although this procedure has been used in illustrations in section 2 for $\theta = \pi/2$, we give sample results for $j = 1$ and 2 for all values of θ. For $j = 1$, the quantum Wigner functions are

$$\begin{aligned}C^1_{0,\lambda(\tilde{m},\theta)}(\theta) &= \frac{2}{\sqrt{[2]}S_q(\theta)}\{\lambda(\tilde{m},\theta)-C_q(\theta)\}C^1_{1,\lambda(\tilde{m},\theta)}(\theta)\\ C^1_{-1,\lambda(\tilde{m},\theta)}(\theta) &= \frac{1}{[2]S_q^2(\theta)}\{4\lambda(\tilde{m},\theta)(\lambda(\tilde{m},\theta-C_q(\theta))-[2]S_q^2(\theta)\}C^1_{1,\lambda(\tilde{m},\theta)}(\theta)\end{aligned} \tag{36}$$

where the various values of $\lambda(\tilde{m},\theta)$ are

$$\lambda(\tilde{m},\theta) = 0 \qquad \lambda(\tilde{m},\theta) = \pm\left\{\frac{[2]}{2}S_q^2(\theta)+C_q^2(\theta)\right\}^{1/2}. \tag{37}$$

For $j = 2$, we find

$$\begin{aligned}C^2_{1,\lambda(\tilde{m},\theta)}(\theta) &= \frac{2(\lambda(\tilde{m},\theta)-2C_q(\theta))}{\sqrt{[4]}S_q(\theta)}C^2_{2,\lambda(\tilde{m},\theta)}(\theta)\\ C^2_{0,\lambda(\tilde{m},\theta)}(\theta) &= \frac{1}{\sqrt{([2][3][4])}S_q^2(\theta)}\{4(\lambda(\tilde{m},\theta)-C_q(\theta))(\lambda(\tilde{m},\theta)-2C_q(\theta))\\ &\quad -[4]S_q^2(\theta)\}C^2_{2,\lambda(\tilde{m},\theta)}(\theta)\end{aligned}$$

$$C^2_{-1,\lambda(\tilde{m},\theta)}(\theta) = \{\frac{2\lambda(\tilde{m},\theta)(4(\lambda(\tilde{m},\theta) - C_q(\theta))(\lambda(\tilde{m},\theta) - 2C_q(\theta)) - [4]S_q^2(\theta))}{[2][3]\sqrt{[4]}S_q^3(\theta)}$$
$$- \frac{2(\lambda(\tilde{m},\theta) - 2C_q(\theta))}{\sqrt{[4]}S_q(\theta)}\}C^2_{2,\lambda(\tilde{m},\theta)}(\theta)$$
$$C^2_{-2,\lambda(\tilde{m},\theta)}(\theta) = \left\{\frac{16\lambda(\tilde{m},\theta)(\lambda(\tilde{m},\theta) + C_q(\theta))(\lambda(\tilde{m},\theta) - C_q(\theta))(\lambda(\tilde{m},\theta) - 2C_q(\theta))}{[2][3][4]S_q^3(\theta)}\right.$$
$$- \frac{4[4]\lambda(\tilde{m},\theta)(\lambda(\tilde{m},\theta + C_q(\theta))S_q^2(\theta)}{[2][3][4]S_q^3(\theta)}$$
$$- \frac{4(\lambda(\tilde{m},\theta) + C_q(\theta))(\lambda(\tilde{m},\theta) - 2C_q(\theta))}{[4]S_q^2(\theta)}$$
$$\left. - \frac{1}{[4]S_q^2(\theta)}\{4(\lambda(\tilde{m},\theta) - C_q(\theta))(\lambda(\tilde{m},\theta) - 2C_q(\theta)) - [4]S_q^2(\theta)\}\right\}$$
$$\times C^2_{2,\lambda(\tilde{m},\theta)}(\theta) \qquad (38)$$

where the values for $\lambda(\tilde{m},\theta)$ can be obtained by equating $C^2_{-3,\lambda(\tilde{m},\theta)}(\theta)$ to zero.

When $q = 1$, we have $[n] = n$ and so $a_m = 2(\tilde{m} - m\cos(\theta))$ and $b_p = p(2j - p + 1)\sin^2\theta$ as the quasic-periodic circular functions go over to the trignometric functions. Then (35) gives

$$\{(p+1)(2j-p)\}^{1/2}\sin^2\theta d^j_{j-(p+1),\tilde{m}}(\theta)$$
$$= \left\{a_{j-p} - \frac{b_p}{a_{j-(p-1)}-}\frac{b_{p-1}}{a_{j-(p-2)}-}\frac{b_{p-2}}{a_{j-(p-3)}-}\cdots\frac{b_1}{a_j}\right\}d^j_{j-p,\tilde{m}}(\theta) \qquad (39)$$

a formula for the $SU(2)$ Wigner d-function ratios as a continued fraction. In order to verify this result, we proceed as follows. In the case of ordinary $SU(2)$, the Wigner d-functions can be expressed in terms of the ${}_2F_1$-hypergeometric function [6]. The ratio,

$$\frac{{}_2F_1(a, b+1; c+1; z)}{{}_2F_1(a, b; c; z)}$$

of two ${}_2F_1$-hypergeometric functions with unit increase in the second and third arguments is the well known continued fraction (Gauss formula [10])

$$\frac{1}{1-}\frac{a(c-b)z/c(c+1)}{1-}\frac{(b+1)(c-a+1)z/c(c+1)(c+2)}{1-}\cdots. \qquad (40)$$

The ratio $d^j_{j-2,\tilde{m}}(\theta)/d^j_{j-1,\tilde{m}}(\theta)$ has been evaluated using (39) and found to reproduce the expression obtained using [6] and (40). The formula for the Wigner d-functions for $q = 1$ in terms of continued fractions is consistent with the property of the ${}_2F_1$-hypergeometric function. So the formula for the Wigner d-function for $q \neq 1$ suggests a generalization of the Gauss formula for a ${}_2\phi_1$ q-hypergeometric function. It is not clear at present that these continued fraction results have physical applicability.

4. Summary

We have explicitly constructed squeezed angular momentum states for $SU_q(2)$ in terms of quantum Wigner C-functions, defined in this paper. These C-functions are analogous to $d^j_{mm'}$ Wigner functions. The squeezed states satisfy the minimum angular momentum uncertainty product with $\triangle J_1 \neq \triangle J_2$. The above construction uses the notion of rotation about the y axis by $\pi/2$. This has been generalized to all values of θ by using quasi-periodic circular functions and an expression for quantum Wigner C-functions as a continued fraction is obtained. For $q = 1$, this gives a relation between $d^j_{mm'}$ and $d^j_{m-1,m'}$ which has been verified. It is suggested that the C-functions are expressible in terms of a ${}_2\phi_1$-q-hypergeometric function, thereby giving an algebraic setting for a q-hypergeometric function in the study of rotation in quantum group $SU_q(2)$.

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